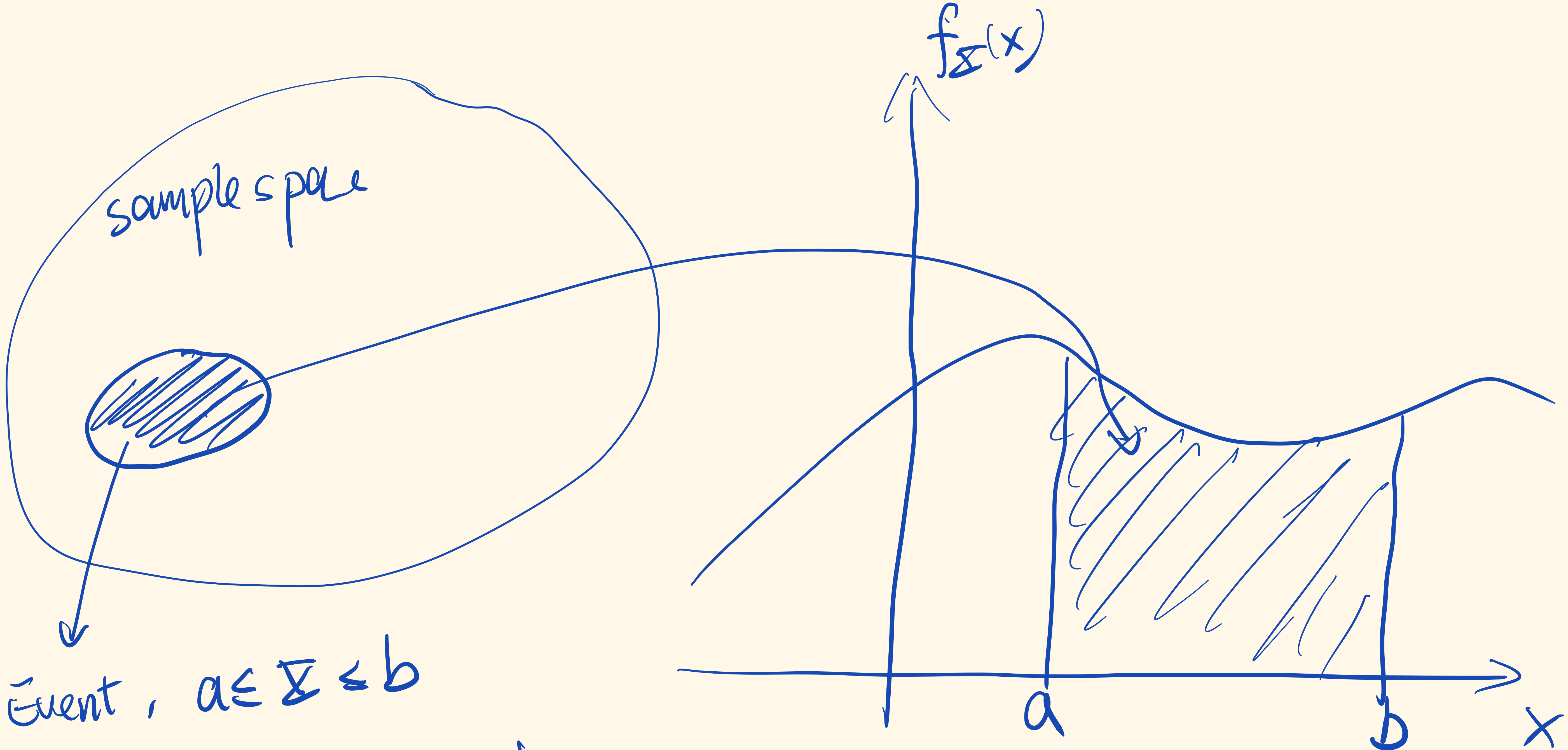
Continuous Random Variable I

Aug 2, 2022

Continuous sample space



$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Probability mass function (PMF)

- For discrete random variable $\boldsymbol{X}$, probability mass function (PMF)denoted as $p_{\boldsymbol{X}}(x) = \mathbb{P}(\boldsymbol{X} = x)$ captures the probabilities of values that $\boldsymbol{X}$ can take.

- $\sum_x p_{\boldsymbol{X}}(x) = 1$

Probability density function (PDF)

- A random variable if called continuous if there is a nonnegative function f_X called <u>probability density function</u> (PDF) of $\boldsymbol{X}$ such that

$$\mathbb{P}(\boldsymbol{X} \in B) = \int_B f_X(x)dx \quad \text{for every subset } B \subset \mathbb{R}.$$

- The probability that the value of $\boldsymbol{X}$ falls with in an interval is

$$\boxed{\mathbb{P}(a \le \boldsymbol{X} \le b) = \int_b^a f_X\,(x)dx}$$

single value in PDF does not map to probability

Probability density function (PDF)

$f_X(x) \geq 0, \ \forall x$

$$\mathbb{P}(a \leq \boldsymbol{X} \leq b) = \int_b^a f_{\boldsymbol{X}}(x)dx$$

- The probability of $\boldsymbol{X}$ taking a single value is 0

$$\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f_X(x)\,dx = 0$$

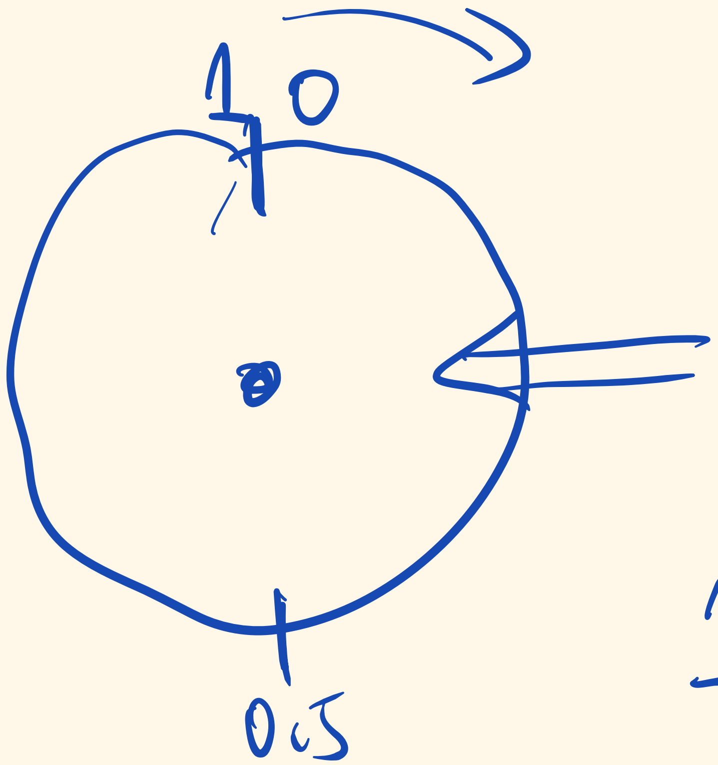
$$\mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X \leq b)$$

- Normalization property

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \mathbb{P}(-\infty < X < \infty) = 1$$

Example 1: Continuous uniform random variable

Spinning a wheel of fortune. The arrow continuously takes value between [0, 1]. Observe the number that the arrow points at.

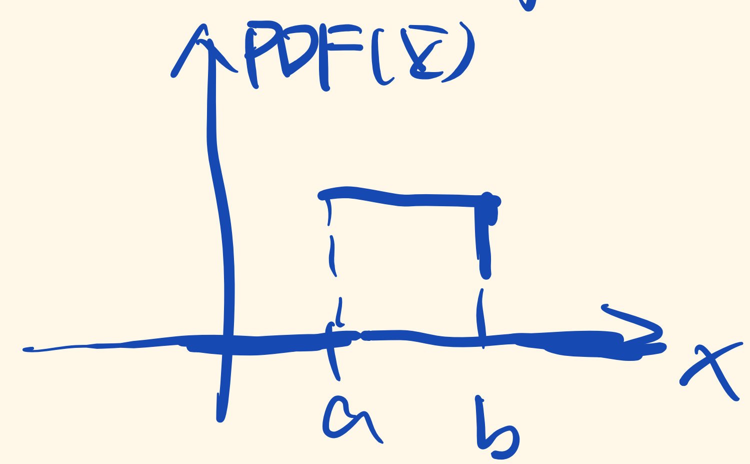


$$f_X(x) = \begin{cases} c & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 c\,dx = c \qquad \therefore c = 1$$

for a general interval $[a, b]$ we refer to R.V. as uniform w/

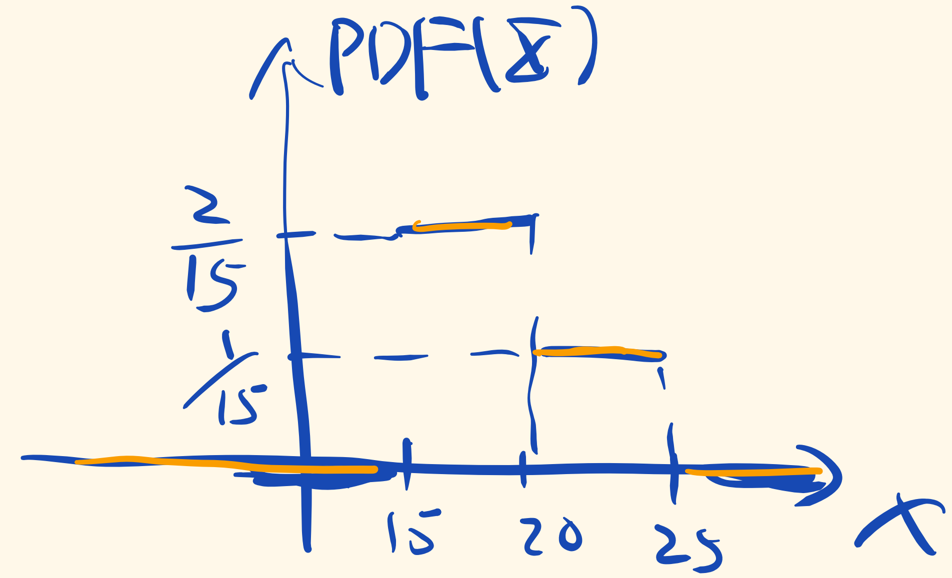
pdf: $$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{o.w.} \end{cases}$$



Example 2: Piecewise constant PDF

Alice walks to class. It takes 15-20 min if it's sunny; it takes 20-25 min if it's rainy. Walking time being equally likely in each case. If in this city, the probability of a day is sunny is 2/3; a day is rainy is 1/3. What's the PDF of walking time ***X***

$$f_X(x) = \begin{cases} c_1 & 15 \le x \le 20 \\ c_2 & 20 < x \le 25 \\ 0 & \text{o.w.} \end{cases}$$



$$\mathbb{P}(\text{sunny}) = \int_{15}^{20} f_X(x)\,dx = 5c_1 = \frac{2}{3}$$

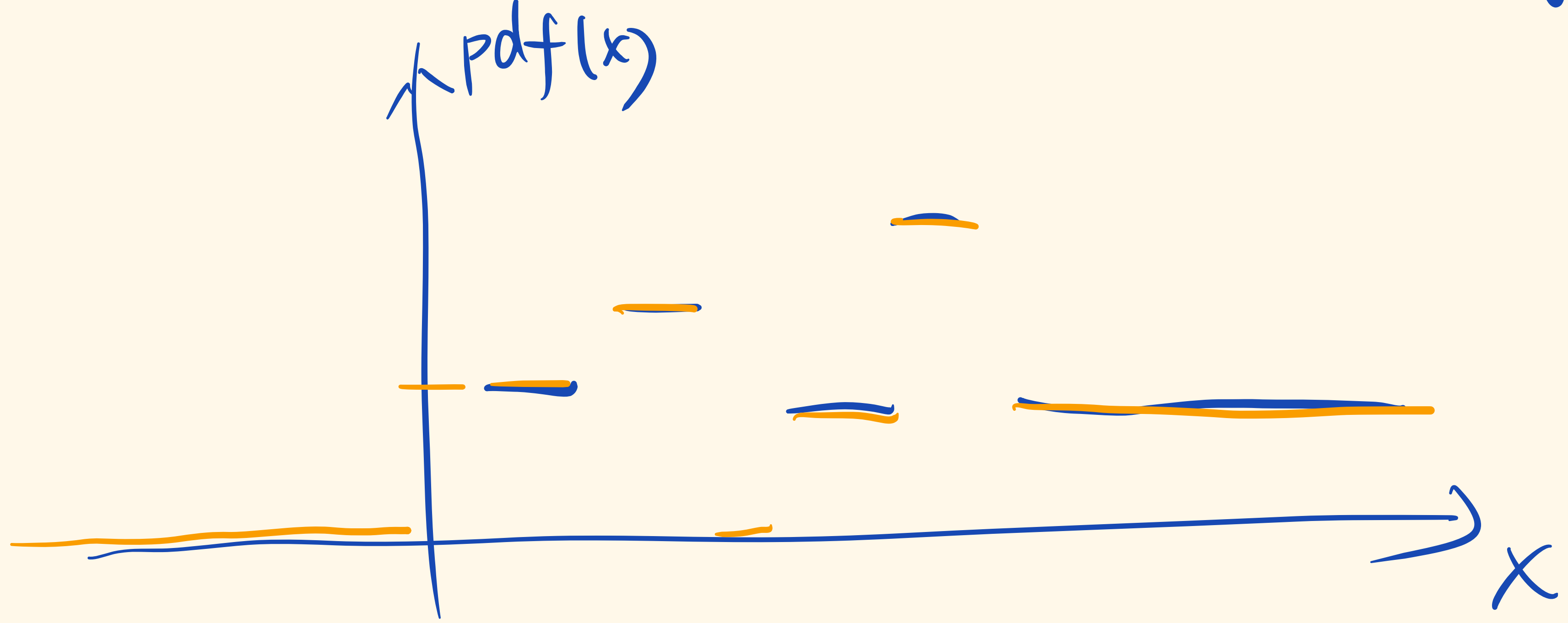
$$\mathbb{P}(\text{rainy}) = \int_{20}^{25} f_X(x)\,dx = 5c_2 = \frac{1}{3}$$

$$\Rightarrow \quad c_1 = \frac{2}{15}, \quad c_2 = \frac{1}{15}.$$

General piecewise constant PDF

$$f_X(x) = \begin{cases} c_i & \text{if } a_i \le x \le a_{i+1} \quad i = 1, \cdots, n-1 \\ 0 & \text{o.w.} \end{cases}$$

then $$1 = \int_{a_1}^{a_n} f_X(x)\,dx = \sum_{i=1}^{n-1} \int_{a_i}^{a_{i+1}} c_i\,dx = \sum_{i=1}^{n-1} c_i (a_{i+1} - a_i)$$



Example 3: A PDF can take arbitrarily large value

Consider a random variable $\boldsymbol{X}$ with PDF

$$f_X(\boldsymbol{x}) = \begin{cases} \dfrac{1}{a\sqrt{x}} & \text{if } 0 < x < 1 \\ 0 & otherwise \end{cases}$$

$\mathbb{P}(X \in (0,1)) = \mathbb{P}(X \in [0,1)) \quad \mathbb{P}(X = 0) = 0$

$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{a\sqrt{x}}\,dx = \frac{2}{a}\sqrt{x}\,\Big|_0^1 = 1$ when $a=2$

$f_X(x)$ when $a=2$ is a valid pdf.

Summary of PDF

- A continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$

non negative

$$f_{\boldsymbol{X}}(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f_{\boldsymbol{X}}(x)dx = 1$$

For $B \subset \mathbb{R}$, $\mathbb{P}(\boldsymbol{X} \in B) = \int_B f_{\boldsymbol{X}}(x)dx$

Expectation

The expected value or expectation or mean of a continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ is defined by

PDF

$$\mathbb{E}(\boldsymbol{X}) = \int_{-\infty}^{\infty} x\, \underline{f_{\boldsymbol{X}}(x)}\, dx$$

Variance

The variance of a continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ is defined by

$$Var(\boldsymbol{X}) = \mathbb{E}\left(\boldsymbol{X}^{2}\right) - \mathbb{E}(\boldsymbol{X})^{2}$$
$$= \int_{-\infty}^{\infty} x^{2} f_{\boldsymbol{X}}(x) dx - \left(\int_{-\infty}^{\infty} x\, f_{\boldsymbol{X}}(x) dx\right)^{2}$$

Example 4: mean and variance of the uniform random variable

Consider a uniform pdf over an interval [a, b]

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \, f_X(x)\, dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \frac{1}{2} x^2 \Big|_a^b = \frac{a+b}{2}$$

$$\mathbb{E}(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \frac{1}{3} x^3 \Big|_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

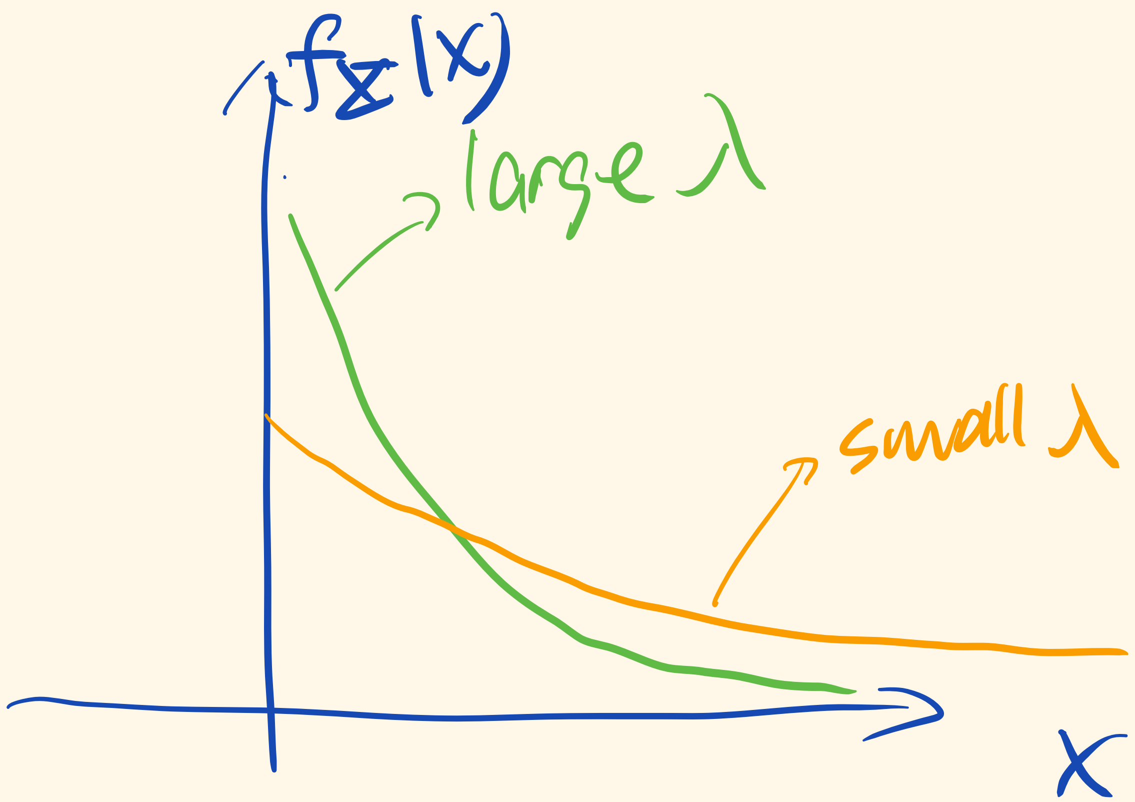
$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \frac{a^2 + ba + b^2}{3} - \frac{a^2 + b^2 + 2ab}{4} = \frac{(b-a)^2}{12}$$

Exponential Random Variable $\lambda > 0$

- An exponential random variable has a PDF of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_0^{\infty} = 1$$

legit PDF.

$$\mathbb{P}(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda x}\,dx = e^{-\lambda x}\Big|_a^{\infty} = e^{-\lambda a}$$

$$X \sim Exp(\lambda) \iff f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

integrate by part

$$\mathbb{E}(X) = \int_0^\infty x \lambda e^{-\lambda x}\, dx = -xe^{-\lambda x} \Big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^\infty = \frac{1}{\lambda}$$

$$\mathbb{E}(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \underbrace{-x^2 e^{-\lambda x} \Big|_0^\infty}_{0} + \int_0^\infty 2x \cdot e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda} \mathbb{E}(X) = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Example 5.

X

Time till a small meteorite first lands anywhere in a desert is modeled as an exponential r.v. with mean of 10 days. It is currently might night, what is the probability that a meteorite first lands between 6am to 6pm of the day?

$\frac{1}{\lambda} = 10$ days

$X \sim Exp(1/10)$

$$\mathbb{P}(1/4 \le X \le 3/4) = \mathbb{P}(X \ge \tfrac{1}{4}) - \mathbb{P}(X > 3/4)$$

$$\Downarrow$$

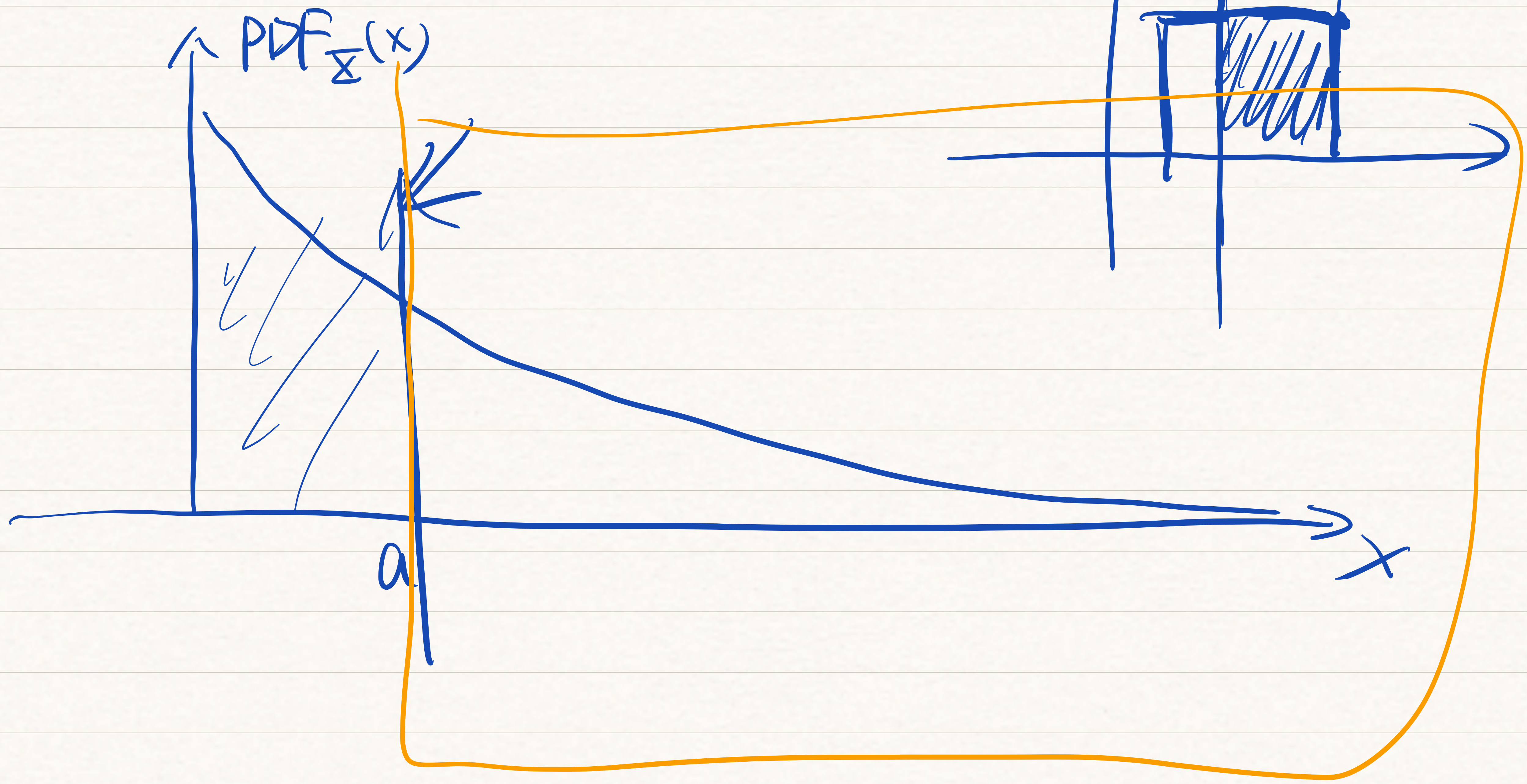
$$\int_{1/4}^{3/4} \frac{1}{10} e^{-\frac{x}{10}} dx$$

$$= e^{-\frac{1}{4}\lambda} - e^{-\frac{3}{4}\lambda}$$

$$= e^{-\frac{1}{40}} - e^{-3/40} = 0.0476$$

$$\boxed{\mathbb{P}(X \ge a) = \mathbb{P}(X > a) = e^{-\lambda a}}$$

$$\mathbb{P}(X \geq a) = e^{-\lambda a}$$



Cumulative density function (CDF)

any random

The CDF of a ~~continuous random~~ variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ is denoted as $F_{\boldsymbol{X}}$

or with PMF $P_X(k)$

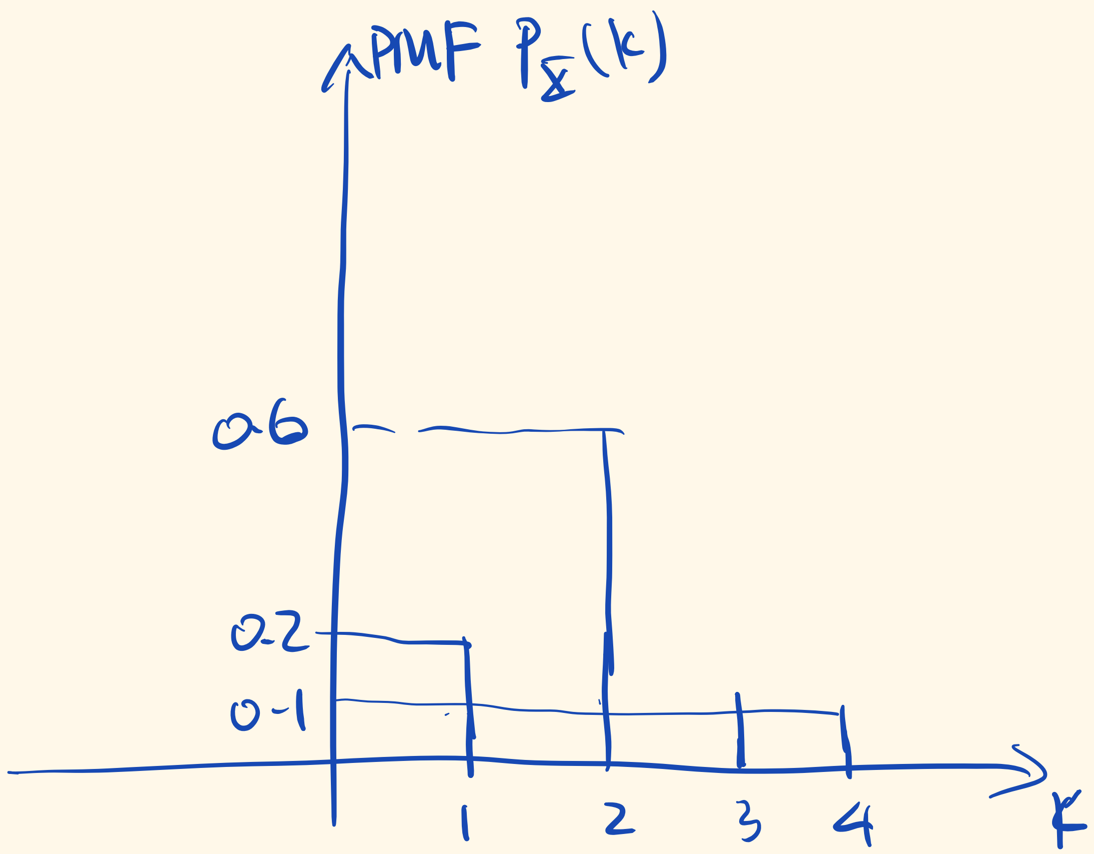
$\forall x,$

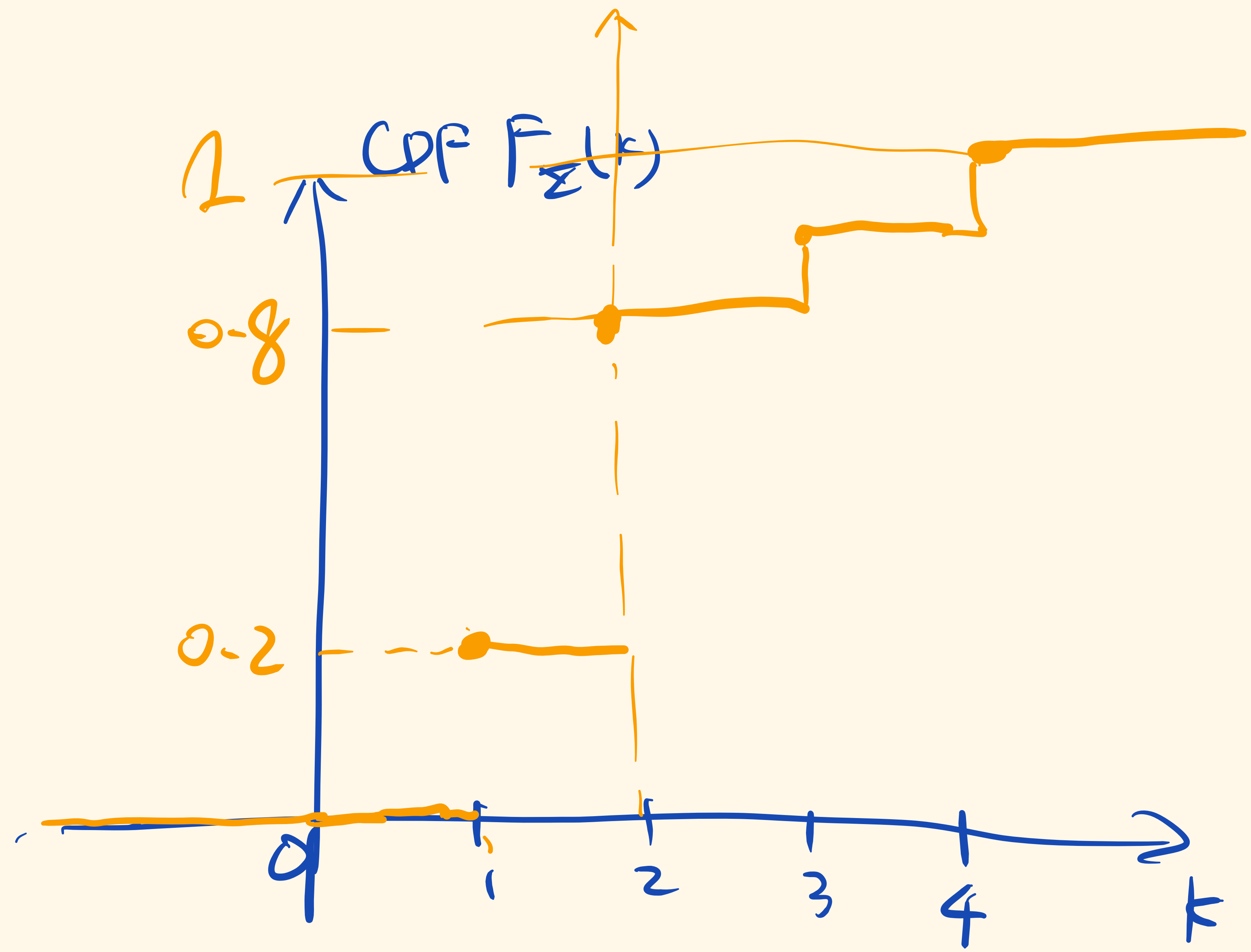
$\leqslant$

$$F_{\boldsymbol{X}}(x) = \mathbb{P}(\boldsymbol{X} \leq x) = \begin{cases} \displaystyle\sum_{k \leq x} p_{\boldsymbol{X}}(k) & \text{if } \boldsymbol{X} \text{ is discrete} \\ \displaystyle\int_{-\infty}^{x} f_{\boldsymbol{X}}(t)dt & \text{if } \boldsymbol{X} \text{ is continuous} \end{cases}$$

Example 1

$\mathbb{P}(X \leq x)$





$$\mathbb{P}(X \leq 2) = \mathbb{P}(X=1) + \mathbb{P}(X=2)$$
$$= 0.2 + 0.6 = 0.8$$

$\to b$

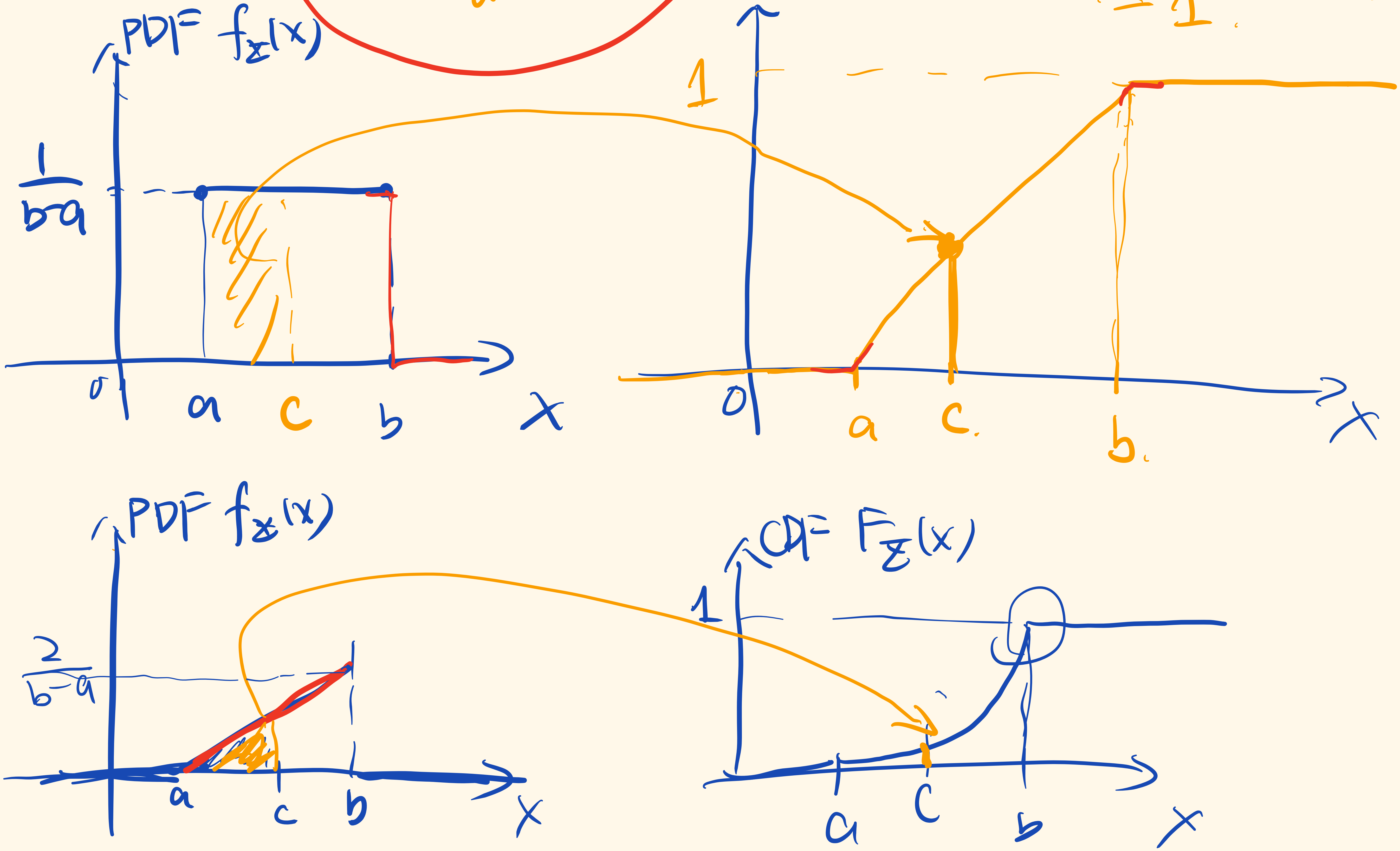
Example 2

$$\mathbb{P}(X \le c) = \int_a^c f_X(x)\,dx$$

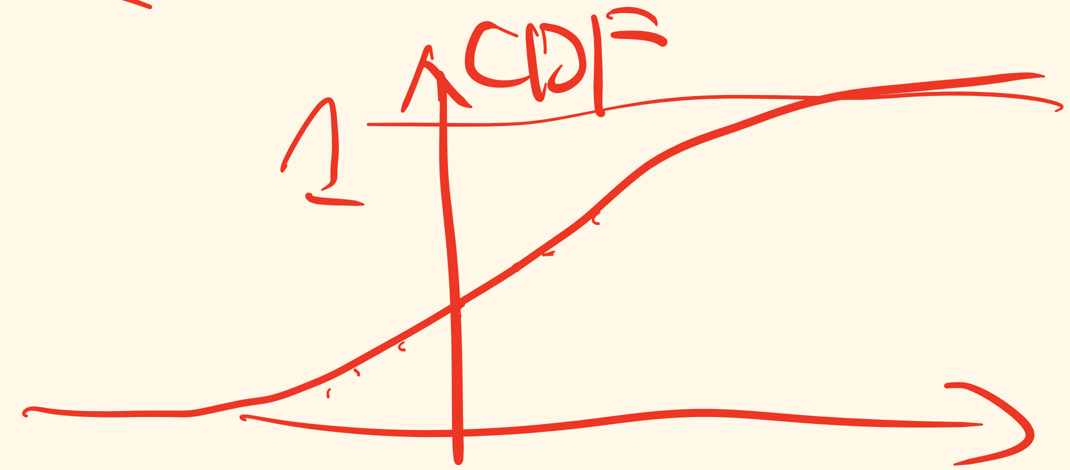
$$\mathbb{P}(X \le x)$$

$$\mathbb{P}(X \le b+1) = \mathbb{P}(X \in (a,b)) = 1$$

CDF $F_X(x)$



Properties of a CDF $F_{\boldsymbol{X}}(x) = \mathbb{P}(\boldsymbol{X} \leq x)$



- $F_{\boldsymbol{X}}(x)$ is monotonically nondecreasing.
  - if $x \leq y$ then $F_{\boldsymbol{X}}(x) \leq F_{\boldsymbol{X}}(y)$.
- $F_{\boldsymbol{X}} \rightarrow 0$ as $x \rightarrow -\infty$, $F_{\boldsymbol{X}} \rightarrow 1$ as $x \rightarrow \infty$.
- If $\boldsymbol{X}$ is discrete then $F_{\boldsymbol{X}}(x)$ is a piecewise constant function of x.
- If $\boldsymbol{X}$ is continuous then $F_{\boldsymbol{X}}(x)$ is a continuous function of x.
- If $\boldsymbol{X}$ is discrete and takes integer values, then PMF and the CDF can be obtained by summing or differencing,
  - $F_{\boldsymbol{X}}(k) = \sum_{i=-\infty}^{k} p_{\boldsymbol{X}}(k)$, $\quad p_{\boldsymbol{X}}(k) = \mathbb{P}(\boldsymbol{X} \leq k) - \mathbb{P}(\boldsymbol{X} \leq k-1) = F_{\boldsymbol{X}}(k) - F_{\boldsymbol{X}}(k - \mathbf{1})$
- If $\boldsymbol{X}$ is continuous, then PDF and the CDF can be obtained by integration or differentiation,
  - $F_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{-\infty}^{x} f_{\boldsymbol{X}}(t)dt$, $\quad f_{\boldsymbol{X}}(x) = \frac{dF_{\boldsymbol{X}}}{dx}(x)$.